Journal of Statistics: Advances in Theory and Applications
Volume 10, Number 2, 2013, Pages 189-197

SOME PROPERTIES OF THE STEADY STATE CHARACTERISTICS OF DISCRETE TIME QUEUEING MODELS

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Abstract

With an aim to identify queueing systems, we investigate the infinite divisibility property of the steady state distribution of certain queue characteristics like queue length, waiting time, number of customers served during the busy period etc. of certain discrete time single/bulk arrival queueing models with general service time distribution.

2010 Mathematics Subject Classification: 60K25.

Keywords and phrases: infinite divisibility, queue length, waiting time, busy period, steady state distributions.

Received September 23, 2013

1. Introduction

The analysis of discrete time queueing models has received considerable attention in the scientific literature over the past years. In computer and telecommunication systems, the basic time unit is a fixed interval called a packet or ATM cell of transmission time. Therefore, discrete time systems are more appropriate for studying computer and telecommunication systems. Also, the current production system of numerous factories operate on a discrete time basis where events can only happen at regularly spaced epochs. The discrete time queueing models have been developed as continuous counterparts (Hunter [1]; Takagi [2]). It is important to have some knowledge of structural aspects of distributions frequently encountered in the theory of queues. The properties, such as infinite divisibility, log-convexity, stability etc. could provide us an insight about the behaviour of the distributions.

The problem of identifiability is of paramount importance in all statistical methods and data analysis and it occurs in all fields, where stochastic modelling is widely used. In many field, the objective of the investigator's interest is not just the population or the probability distribution of an observable random variable, but the physical structure or the model leading to the probability distribution. Identification problems arise when observations can be explained in terms of one of several available models. Any statistical procedure developed based on a stochastic model is meaningful only if the model is identifiable. Manoharan et al. [3] bring together the revelent materials on identifiability problems in diverse fields of queueing theory, such as network, telecommunications etc.. In identification of queueing models, the knowledge about the structural aspects of distributions frequently encountered is very important. The structural properties, such as unimodality, infinite divisibility, log-convexity, stability etc. could provide us an insight about the behaviour of the distributions.

Many of the well-known distributions belong to the class of infinitely divisible distributions. Infinite divisibility is one of the most important concepts in distribution theory. These properties as well as the related topics like log convexity, self decomposability, stability etc. are of potential importance in both theoretical and practical problems in statistics and allied fields. Indeed, these properties determine certain structural aspects of probability distribution. If the properties in question identify certain models fully or partially, one could utilize them with advantage in a model building problem or in any problem that is concerned with the analysis of the characteristics of a model.

Infinite divisibility and related properties of the distributions of different queue characteristics of continuous time queueing models have been studied by several researchers in the past. Kingman [4] showed that for a G/G/1 queue with traffic intensity $\rho < 1,$ the steady state waiting time distribution is compound geometric and hence infinitely divisible. Burke [5] proved that for M/M/s queueing system with infinite waiting space, the steady state distribution of the number of departures during an interval is Poisson and hence infinitely divisible. Mirasol [6] showed that the steady state distributions of the number of departures during an interval of M/M/1 queueing system is also Poisson and hence infinitely divisible. Shanbhag and Tambouratzis [7] have established that in the system M/G/s with waiting space $N = 0,$ if the losses are included in the departures, then the steady state distribution of the number of departures during an interval are Poisson and infinitely divisible. Shanbhag [8] characterizes the M/G/1 queue amongst M/G/s/N queueing system via infinite divisibility of the joint distribution of the number of arrivals and departures during any fixed interval. Manoharan et al. [3] extended this result to the case of GI/G/s/N system. They also show that the queue length distribution of an $\mathrm{M/M/s}\,(1 \leq s \leq \infty)$ queueing system is infinitely divisible if, and only if, (iff) $s = 1$ or 2 or $\infty.$ Recently, Jose and Manoharan [9] considered the infinite divisibility and related properties of the steady state distribution of certain continuous time queue characteristics.

A detailed discussion on the properties of infinitely divisible distributions can be found in Steutel and Van Harn [10].

In this paper, we investigate the infinite divisibility and related properties of the steady state distribution of certain queue characteristics like queue length, waiting time, number of customers served during the busy period etc. of various discrete time queueing systems. In Section 2, we deal with the characteristics of Geo/G/1 queueing system. In Section 3, we discuss the characteristics of Geo/G/1 queueing system with vacation.

2. Geo/G/1 Queueing Model

Consider the classical discrete time Geo/G/1 queueing system. In this system, we assume that customer arrivals can only occur at discrete time instants $t = n^-$, $n = 0, 1, 2, \cdots$. The service starting and ending time can only occur at discrete time instants $t = n^+$, $n = 0, 1, 2, \cdots$. The arrival times are iid discrete random variables, denoted by T, with a geometric distribution of parameter p. That is $P[T = j] = p(\overline{p})^{j-1}$, $j = 1, 2, 3, \ldots$, where $\overline{p} = 1 - p$. Thus, the number of arrivals in the interval $[0, n]$, $C_n \sim B(n, p)$. The service times are also iid, discrete random variables denoted by S, with a general distribution $P[S = j] = g_j$, $j \geq 1$ and PGF, $G(z) = \sum_{j=1}^{\infty} z^j g_j$.

We assume that inter arrival times and the service times are independent and the service order is FCFS. Let L be the stationary queue length and W be the stationary waiting time. For $\rho < 1$, like the Pollaczek-Khinthin formula for the continuous time M/G/1 system, we have

$$L(z) = \frac{(1-\rho)(1-z)G(1-p+zp)}{G(1-p+zp)-z},$$

$$W(z) = \frac{(1-\rho)(1-z)}{(1-z)-\rho(1-G(z))}.$$

Theorem 2.1. *In a Geo/G/1 queueing system, the steady state system size distribution is infinitely divisible, if the service time distribution is infinitely divisible.*

Proof. Let A be the number of arrivals during the service time of a unit with $k_j = P[A = j]$. Then we have $A(z) = G(1-p+pz)$ and $A(z)$ is infinitely divisible if $G(z)$ is infinitely divisible.

Define $g_n = \dfrac{(1-\sum_{j=0}^{n} k_j)}{\rho}$, $n = 0, 1, 2, \cdots,$ then $G(z) = \sum_{n=0}^{\infty} g_n z^n = \dfrac{A(z)-1}{\rho(z-1)}$. Let $q_n = \sum_{m=0}^{\infty} (1-\rho)\rho^m g_n^{(m)^*}$, where $g_n^{(m)^*}$, be the coefficient of z^n in $[G(z)]^m$. Then the PGF of (q_n) is $Q(z) = \sum_{n=0}^{\infty} q_n z^n = \dfrac{1-\rho}{1-\rho G(z)}$. $Q(z)$ is the PGF of a compound geometric distribution and hence infinitely divisible. Moreover $L(z) = Q(z)A(z)$. Hence $L(z)$ is infinitely divisible.

Distribution of the number of customers served during the busy period

Let T be the busy period, then $T = S + T_1 + T_2 + \cdots + T_A$, where S is the service time of the initiating customer and A is the number of arrivals during the service time of the initial customer T_i's are iid as T with PGF $D(z)$.

Then

$$E(z^T / S = k) = \sum_{j=0}^{k} E(z^T / S = k,\ A = j),$$

$$D(z) = E(z^T) = \sum_{k=0}^{\infty} E(z / S = k) P(S = k)$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} E(z^T / S = k,\ A = j) \binom{k}{j} p^j (1-p)^{k-j} P(S = k)$$

$$= \sum_{k=0}^{\infty} z^k g_k \sum_{j=0}^{k} (D(z))^j \binom{k}{j} p^j (1-p)^{k-j}$$

$$= \sum_{k=0}^{\infty} z^k g_k (1 - p + pD(z))^k$$

$$= G(z(1 - p + pD(z))).$$

Also, let N be the number of customers served during the busy period and $P(z)$ be its PGF, then we have $N = 1 + N_1 + N_2 + \cdots + N_A$, where N_i is the number of customers arrived during the service time of the i-th customer arrived during the service time of the initiating customer.

Then we have

$$E(z^N / A = j) = E(z) \prod_{i=i}^{j} P(z) = z(P(z))^j,$$

$$P(z) = E(z^N) = \sum_{j=0}^{\infty} E(z^N / A = j) P(A = j)$$

$$= z \sum_{j=0}^{\infty} (P(z))^j P(A = j)$$

$$= zG(1 - p + pD(z)).$$

Theorem 2.2. *The number of customers served during the busy period of a Geo/G/*1 *queueing system is infinitely divisible, if and only if the service time distribution is infinitely divisible.*

Proof. We have if $G(z)$ is infinitely divisible and $0 < p < 1,$ then $G(1 - p + pz)$ is infinitely divisible. Also, the composition of a infinitely divisible PGF $G(z)$ with a PGF $P(z), G \circ P,$ is infinitely divisible. Therefore, $G(1 - p + pP(z))$ and hence $zG(1 - p + pP(z))$ are infinitely divisible if the service time distribution is infinitely divisible.

Conversely, suppose that $P(z)$ is infinitely divisible. Then $\frac{P(z)}{z}$ is infinitely divisible and hence service time distribution is infinitely divisible.

Theorem 2.3. *Let* $P(z)$ *be PGF of the number of customers served during the busy period of a Geo/G/*1 *queueing system. Then* $\frac{P(z)}{z}$ *is compound geometric, if and only if the service time distribution is geometric.*

Proof. Suppose that the service time distribution is geometric with parameter q, then we have

$$\frac{P(z)}{z} = G(1 - p + P(z)) = \frac{1 - q}{1 - q(1 - p + P(z))} = \frac{1 - c}{1 - cP(z)}, \text{ where } c = \frac{pq}{1 - q^2}.$$

Conversely, assume that $\frac{P(z)}{z}$ is compound geometric.

Hence

$$\frac{P(z)}{z} = G(1 - p + P(z)) = \frac{1 - c}{1 - cP(z)}.$$

Take $c = \frac{pq}{1 - q^2}$, $0 < p, q < 1.$ Then we have $\frac{P(z)}{z} = \frac{1 - q}{1 - q(1 - p + P(z))}$.

Hence $G(z) = \frac{1 - q}{1 - qz}$, which is the PGF of the geometric distribution.

Remark 2.4. Clearly from the above theorem, $A(z) = G(1 - p + pz)$ is compound geometric iff the service time distribution is geometric. Hence from Equation (2.9), the steady state distribution of the system size of a Geo/G/1 queueing system can be represented as the product of two compound geometric distributions iff the service time distribution is geometric.

3. $\text{Geo}^{\text{X}}/\text{G}/1$ Queueing Model

So far we have considered single arrival discrete time queueing system. Now, we assume that customers arrive in batches of random size C, where C has the distribution $c_n = P[C = n]$; $n \geq 1$ and the PGF $C(z) = \sum_{n=1}^{\infty} c_n z^n$.

The PGF of the steady state system size distribution of the bulk arrival queueing systems with the general service time distribution is $L(z) = \dfrac{(1 - \rho)(1 - z)G(1 - p + pC(z))}{G(1 - p + pC(z)) - z}$.

Theorem 3.1. *In a* $Geo^X/G/1$ *queueing system, the steady state system size distribution is infinitely divisible if the service time distribution is infinitely divisible.*

Proof. By the properties of infinitely divisible distributions $A(z) = B(1 - p(1 - C(z)))$ is an infinitely divisible PGF, if the service time distribution is infinitely divisible. Now proceeding as in the proof of Theorem 2.1, we can show that $P(z) = Q(z)A(z)$ and $P(z)$ is an infinitely divisible PGF, if the service time distribution is infinitely divisible.

4. Conclusion

In identification of queueing models, the knowledge about the structural aspects of distributions frequently encountered there is very important. The structural properties such as infinite divisibility, log-convexity, stability etc. could provide us an insight about the behaviour of the distributions. In this paper, we established some structural properties of the steady state characteristics of various discrete time queueing models.

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